# METRICALLY HOMOGENEOUS SETS: CORRIGENDUM

BY

BRANKO GRÜNBAUM AND L. M. KELLY

ABSTRACT

A flaw in Lemma 2 is corrected and other parts of the paper changed accordingly so as to preserve all the theorems.

In reviewing our paper [1] Professor J. F. Rigby has observed that Lemma 2 is not valid as it stands.* The lemma can be repaired by adding the hypothesis $\measuredangle ABC > 120°$, but then additional arguments are required to complete the discussion of Case V in the proof of Theorem 6. In the original text Lemmas 2 and 6 were designed to facilitate the exposition contained in the last two paragraphs of the arguments of that case. We find it advantageous to present modified versions of Lemmas 2 and 6, and to revamp correspondingly the argument in Case V of the proof of Theorem 6.

We note also that the reference to Lemma 2 on page 191, line 31, should be* to Lemma 1, and that in various references throughout the paper "Comparison Lemma" is another term for Lemma 1.

LEMMA 2′. *Let the convex quadrilaterals $ABCD$ and $A'B'C'D'$, each labelled in cyclic order, satisfy*

$$AB = CD = B'C' = x, \; A'B' = C'D' = BC = y, \; AC = BD = A'C' = B'D',$$

$$AD = w, \; A'D' = w', \text{ while } \measuredangle ABC > 120° \textit{ and } y > x; \textit{ then } w' > w.$$

PROOF. Noting that $\alpha = 180° - \measuredangle ABC < 60°$ we have

$$w = 2x\cos\alpha + y, \quad w' = 2y\cos\alpha + x$$

$$w' - w = 2\cos\alpha(y - x) + (x - y) = (y - x)[2\cos\alpha - 1] > 0.$$

* We are indebted to Professor Rigby for kindly bringing these facts to our attention.

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LEMMA 6′. *If a homogeneous polygon contains consecutive vertices $A, B, C, D, E$ with $AB = CD = a_1$, $BC = DE = a_2$, $AC = BD = a_3$ and $AD > a_i$, $i = 1,2,3$, and if $\sphericalangle ABC \leqq 120°$, then $AD$ is a diameter of the polygon.*

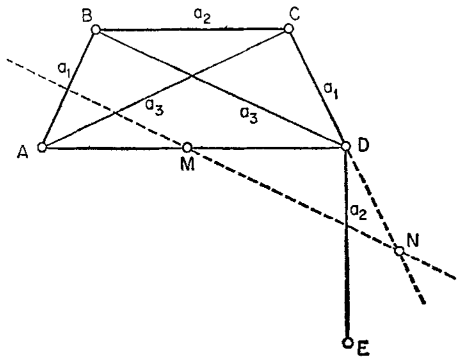


Fig. 1

PROOF. (See Fig. 1.) Let the perpendicular bisector of $AB$ meet $AD$ in $M$ and $CD$ in $N$. Since $AD > BD$, $M$ is between $A$ and $D$. We now claim that $a_2 \geqq MD \geqq ND$. In fact, $\sphericalangle ABM = \sphericalangle BAM \geqq 60°$ and hence $\sphericalangle CBM \leqq 60°$. Thus $\sphericalangle CBM + \sphericalangle BCD \leqq 180$ and $a_2 \geqq MD$.

Furthermore, $\sphericalangle BAM \geqq 60°$ implies that $\sphericalangle DMN \leqq 30°$ which together with $\sphericalangle CDM \geqq 60°$ leads to the conclusion that $DNM \geqq 30°$. Thus $MD \geqq ND$.

It is now clear that $E$ cannot be in the triangle $MDN$ and hence is in the open half plane, $\pi^+$, having line $MN$ as edge and containing point $A$. By the same argument $\pi^+$ contains all the vertices of the polygon with the exception of $B$, $C$ and $D$.

Since $BP > AP$ for all points $P$ in $\pi^+$, the diameter from $A$ must be $AD$. ∎

The last two paragraphs of the argument for case V of Theorem 6 should be replaced by the following (compare Fig. 2): We may suppose $P_0P_2 = P_1P_3 = a_3$.

CASE 1. $\sphericalangle P_0P_1P_2 \leqq 120°$ or $\alpha = 180° - \sphericalangle P_0P_1P_2 \geqq 60°$.

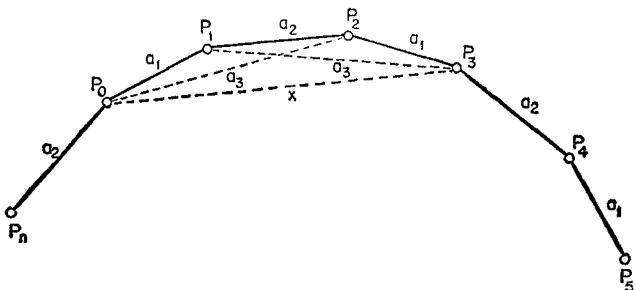


Fig. 2

By the comparison lemma $P_2P_5 \geqq x$. Lemma 6′ implies that $P_0P_3$ is a diameter of the polygon and hence $P_2P_5 = x$, $P_2P_4 = P_3P_5 = a_3$. It can now be inferred by induction that all second order diagonals are of equal length and the polygon is cyclic.

However it is a little bit more revealing to note that, since diametral segments must intersect, $P_4P_1 = x$. Then from quadrilaterals $P_0P_1P_2P_3$ and $P_2P_3P_4P_5$ we conclude that $x = 2a_1 \cos\alpha + a_2 = 2a_2 \cos\alpha + a_1$ and that $\alpha = 60°$. Now the polygonal angles at $P_1, P_2, P_3, P_4$ are all 120° and an easy calculation shows that $P_0P_5 = a_2$. Thus the polygon is a quasiregular hexagon.

CASE 2. $\sphericalangle P_0P_1P_2 > 120°$.

Lemmas 2′ and 1 show that $P_2P_n > x$. If $P_2P_4 = x$ then $a_1 + a_2 \gtreqqless x$ and from Lemma 3 it follows that $\sphericalangle P_0P_1P_2 < 120°$. Since $P_2P_5 \geqq x$ by the comparison lemma, $P_2P_5 = x$. Thus $P_2P_4 = P_3P_5 = a_3$. As above it follows by induction that all second order diagonals are of length $a_3$ and the polygon is cyclic.

## REFERENCE

1. B. Grünbaum and L. M. Kelly, *Metrically homogeneous sets*. Israel J. Math., **6** (1968), 183–197.

MICHIGAN STATE UNIVERSITY
EAST LANSING, MCHIGAN